§5 PARALLEL TRANSPORT IN LINE BUNDLES

We introduce and study the parallel transport induced by a connection on a line bundle.

Let $\pi : L \to M$ a line bundle with a connection ∇.

(5.1) <u>DEFINITION:</u> 1° A HORIZONTAL (or PARALLEL) LIFT of a tangent vector $X \in T_a M$ at $\ell \in L_a = \pi^{-1}(a) \in L^\times$ is a tangent vector $\hat{X} \in T_\ell L$ with

i) $T_\ell \pi(\hat{X}) = X$ ($\hat{X}$ is a LIFT)
ii) $\hat{X} \in H_\ell$ ($\hat{X}$ is HORIZONTAL)

2° Let γ be a (smooth) curve $\gamma : I \to M$ in M ($I \subset \mathbb{R}$ an open intervall). A HORIZONTAL LIFT of γ (through $\ell_0 \in L_{\gamma(t_0)}$) is a smooth curve $\lambda : I \to L$ (with $\gamma(t_0) = \ell_0$) such that

i) $\gamma = \pi \circ \lambda$ (λ is a lift (through ℓ_0)), and
ii) $\dot{\lambda}(t) \in H_{\lambda(t)}$ for all $t \in I$.

In other words: λ is a horizontal lift of γ if λ is a lift and every tangent vector $\dot{\lambda}(t)$, $t \in I$, is horizontal.

[A remark on the notation $\dot{\gamma}(t)$ seems to be appropriate: $\dot{\gamma}(t)$ is the tangent vector at the point $\gamma(t) = a \in M$ given by the curve $s \mapsto \gamma(t+s)$, i.e. $\dot{\gamma}(t) = [\gamma(t+s)]_a \in T_a M$.
Also, with $1 \in T_t I \cong \mathbb{R}$: $\dot{\gamma}(t) = T_t\gamma(1) \in T_a M$]

In order to understand the definition the notion of the horizontal subspace $H_\ell \subset TL^\times$ belonging to the connection ∇ on L will be explained again (see §4 in a general context): For a point $a \in M$ and $\ell \in L_a^\times$ we have a trivialization

$$\varphi : L_U \longrightarrow U \times L$$

of the line bundle $L_U = \pi^{-1}(U) \to U$ over an open neighbourhood of a. On this trivalization the connection ∇ has the form

$$\nabla_X f s_1 = \left(L_X f + 2\pi i \alpha(X) f\right) s_1 \,, \quad f \in \mathcal{E}(U),\ X \in \mathcal{V}(U),$$

with $s_1(a) := \varphi^{-1}(a,1)$ and $\alpha \in \Omega^1(U)$ a one form, the local gauge potential, uniquely defined by ∇: $\alpha(X) \in \mathcal{E}(U)$ is defined by $\nabla_X s_1 = 2\pi i \alpha(X) s_1$. The horizontal space H_ℓ is now given by

$$H_\ell := \left\{ Y = T_{\varphi(\ell)}\varphi^{-1}(X,Z) \in T_\ell L \;\middle|\; X \in T_a U,\ Z \in T_w \mathbb{C} : \tfrac{Z}{w} + 2\pi i \alpha(X) = 0 \right\},$$

$\varphi(\ell) = (a, w) \in U \times \mathbb{C}^\times$.

If $q^1, \dots, q^n$ are local coordinates in U around a then the $Y_j = T_{\varphi(\ell)}\varphi^{-1}\left(\frac{\partial}{\partial q^j}, -2\pi i w \alpha_j\right)$ span H_ℓ.

This digression shows that every $X \in T_a M$ has a unique horizontal lift $\hat{X} \in T_\ell M$ through $\ell \in L_a^\times$ (and the map $\Gamma : T_a M \to H_\ell$ $(\pi(\ell) = a)$ can be used to define a connection - it is the so called EHRESMANN CONNECTION [*])

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* Übung: Find the conditions for Γ to yield a connection.

Moreover,

(5.2) PROPOSITION: Let ∇ be a connection on the line bundle $L \to M$, and let $\gamma : I \to M$ be a (smooth) curve $\gamma(t_0) = a$. For every point $\ell \in L_a^\times$ there exists a uniquely defined horizontal lift $\hat{\gamma} : I \to L^\times$ through ℓ: $\hat{\gamma}(t_0) = \ell$.

□ Proof. In the above local situation one looks for a curve $\xi : I \to \mathbb{C}^\times$ such that $\varphi(\ell) = (a, \xi(t_0))$ and $\hat{\gamma} = \varphi^{-1}(\gamma, \xi)$ is a lift with $\hat{\gamma}(t_0) = \varphi^{-1}(\gamma(t_0), \xi(t_0)) = \ell$. In order that $\hat{\gamma}$ is, moreover, horizontal it has to satisfy

$$2\pi i \alpha(\dot{\gamma}(t)) + \frac{\dot{\xi}(t)}{\xi(t)} = 0,$$

which amounts to the differential equation

$$\dot{\xi}(t) = -2\pi i \alpha(\dot{\gamma}(t))\, \xi(t).$$

And this differential equation has a unique solution on I with $\xi(t_0) \in \mathbb{C}^\times$. □

(5.3) REMARK: From the proof of the proposition we obtain the following characterization: A lift ξ of γ is horizontal if and only if locally

$$\boxed{\dot{\xi}(t) + 2\pi i \alpha(\dot{\gamma}(t))\, \xi(t) = 0,}$$

or – in a very short form – $\nabla_{\dot{\gamma}} \xi = 0$.

This observation allows it to extend the lifting through all points of the fibre, i.e. also through $\ell \in L \setminus L^{\times}$.

Definitions and results extend immediately to connections on a vector bundle E. Such a connection ∇ is locally given by

$$\nabla_X \psi = L_X \psi + \alpha(X).\psi\,, \quad X \in \mathcal{V}(U),\ \psi \in \mathcal{E}(U, \mathbb{K}^r),$$

where $\alpha \in \Omega^1(U, \mathrm{End}(\mathbb{K}^r))$ is a $\mathfrak{g} = \mathrm{End}(\mathbb{K}^r)$-valued 1-form. Hence a horizontal lift of $\gamma : I \to M$, $\gamma(t_0) = a$, looks locally like $\tilde{\gamma} = \varphi^{-1}(\gamma, \eta)$, with $\eta \in \mathcal{E}(I, \mathbb{K}^r)$ and

$$\dot{\eta} + \alpha(\dot{\gamma})\,\eta = 0.$$

Proposition (5.2) leads to the concept of „parallel transport":

(5.4) <u>DEFINITION</u>: With the notation of the last proposition and the choice of $t_1 \in I$ let $\hat{\gamma} = \hat{\gamma}_\ell$ be the horizontal lift of γ with $\hat{\gamma}(t_0) = \ell$. Then the map

$$\ell \mapsto \hat{\gamma}_\ell(t_1)\,, \quad L_{\gamma(t_0)} \to L_{\gamma(t_1)},$$

is an isomorphism (of $\mathbb{C}$ vector spaces). This map is called PARALLEL TRANSPORT ALONG γ and will be denoted by

$$\mathbb{P}^{\gamma}_{t_0,t_1} : L_{\gamma(t_0)} \longrightarrow L_{\gamma(t_1)}\,.$$

The parallel transport $\mathbb{P}^{\gamma}_{t_0,t_1}$ describes a shift of vectors over $\gamma(t_0)$ to those over $\gamma(t_1)$. This shift depends in general on the curve from $\gamma(t_0)$ to $\gamma(t_1)$ (see below). The operators have many interesting properties like

$$\mathbb{P}^{\gamma}_{t_0,t_1} \circ \mathbb{P}^{\gamma}_{t_1,t_0} = id_{L_{\gamma(t_1)}} \quad \text{or}$$

$$\mathbb{P}^{\gamma}_{r,s} \circ \mathbb{P}^{\gamma}_{s,t} = \mathbb{P}^{\gamma}_{r,t} \quad \text{for } r,s,t \in I.$$

One can reconstruct the connection ∇ from the family $\left(\mathbb{P}^{\gamma}_{t_0,t_1}\right)_{\gamma,t_0,t_1}$.

(5.5) <u>DEFINITION:</u> A section $s \in \Gamma(\mathcal{U}, L^{\times})$ over an open subset $\mathcal{U} \subset M$ is called HORIZONTAL if

$$T_a s(T_a M) \subset H_{s(a)} \subset T_{s(a)} L^{\times}$$

holds for all $a \in \mathcal{U}$.

In case of a horizontal section $s \in \Gamma(\mathcal{U}, L^{\times})$ one even has $T_a s(T_a M) = H_{s(a)}$, and $T_a s$ is the inverse of the restriction $T_{s(a)}\pi|_{H_{s(a)}} : H_{s(a)} \to T_a M$ for all $a \in \mathcal{U}$.

$T_a s(T_a M) \subset H_{s(a)}$ implies that each curve $\gamma : I \to \mathcal{U}$, $\gamma(0) = a$, satisfies $(s\circ\gamma)^{\cdot} = T_{\gamma(t)} s(\dot{\gamma}(t)) \in H_{s(\gamma(t))}$, i.e. $s\circ\gamma$ is a horizontal lift of γ. Hence, with $s\circ\gamma = \varphi^{-1}(\gamma, \xi)$ in a local trivialization $\varphi : \mathcal{U}' \to \mathcal{U}' \times \mathbb{C}^{\times}$: $\xi(t) = pr_2\, \varphi(s\circ\gamma(t))$ satisfies

$$\dot{\xi} + 2\pi i \alpha(\dot{\gamma})\xi = 0,$$

and we conclude that $\nabla_X s = 0$ for all $X \in \mathcal{V}(U)$. We have essentially shown:

(5.6) PROPOSITION: Let $L \to M$ be a line bundle with connection. $s \in \Gamma(U,L)$ is horizontal if and only if $\nabla_X s = 0$ for all $X \in \mathcal{V}(U)$.

(5.5) EXAMPLES: 1° In the trival case $L = M \times \mathbb{C}$ and $\alpha = 0$, i.e. $\nabla_X f s_1 = L_X f\, s_1$, we obtain: $s = f s_1$ is horizontal iff f (and hence s) is locally constant.

2° Again in the trivial case $L = M \times \mathbb{C}$ with $M = \mathbb{R}^2$ and $\alpha = q^2 dq^1 - q^1 dq^2$. If $s(a) = (a, f(a))$, $a \in U$, would be a horizontal section with $f(a_0) \neq 0$ at one point $a_0 \in U$ we can assume $f(a) \neq 0$ throughout U (by possibly taking a smaller neighbourhood of a_0). The proposition (5.4) implies $\nabla_X s = 0$, i.e. $L_X f + 2\pi i \alpha(X) f = 0$. Hence,

$$\frac{\partial f}{\partial q^1} + 2\pi i\, \alpha_1 f = \frac{\partial f}{\partial q^1} + 2\pi i\, q^2 = 0,$$

$$\frac{\partial f}{\partial q^2} + 2\pi i\, \alpha_2 f = \frac{\partial f}{\partial q^2} - 2\pi i\, q^1 = 0,$$

and this leads to the contradiction

$$-2\pi i = + \frac{\partial^2 f}{\partial q^1 \partial q^2} = 2\pi i.$$

One can prove the following direct relation between ∇ and the corresponding parallel transport:

$$\nabla_X s(a) = \lim_{r \to 0} \frac{1}{r}\Big(\mathbb{P}_{t+h,t}\big(\, s\circ\gamma(t+h) - s\circ\gamma(t)\big)\Big),$$

where $X = \dot{\gamma}(t) = [\gamma]_a$, $\gamma(t) = 0$.
Therefore, the covariant derivative ∇_X measures along the curve γ to what extent the section s deviates infinitesimally from being horizontal.

Under which conditions does there exist a horizontal section, at least locally? We have seen, that in case of a horizontal section $s \in \Gamma(U, L^\times)$ for each curve γ in U its horizontal lift through $s(\gamma(t_0))$ has the form $s\circ\gamma$. Consequently, for any two points $a, b \in U$ and any curve γ in U with $\gamma(t_0) = a$, $\gamma(t_1) = b$, parallel transport of $l = s(a) = s(\gamma(t_0)) \in L_a$ to L_b along γ is $s\circ\gamma(t_1) = s(b)$: $\mathbb{P}^{\gamma}_{t_0,t_1}(s(a)) = s(b)$ independently of γ (as long as the curves stay in U). For $l' \in L_a^\times$, $l' = cl$, with $c \in \mathbb{C}^\times$, and $s' = cs$ is a horizontal section transporting l' to $cs(b)$, again independently of the curve. We have shown one direction of the following equivalence.

(5.7) <u>PROPOSITION</u>: Let $L \to M$ be a line bundle with connection ∇ and $U \subset M$ open. Then U admits a horizontal section $s \in \Gamma(U, L^\times)$ if and only if the parallel transport from a point $a \in U$ to $b \in U$ is independent of the curves connecting a and b.

$\square$ Proof. Assume that parallel transport is independent of the curves. Without loss of generality we assume furthermore, that U is connected. We obtain to each $a \in U$ and $\ell \in L^X_a$ a unique horizontal section $s : U \to L^X$ with $s(a) = \ell$ by the following prescription: $s(b) := P^{\gamma}_{t_0,t_1}(\ell)$, where γ is a curve $\gamma : I \to U$ with $\gamma(t_0) = a$ and $\gamma(t_1) = b$: $s(b)$ is well-defined since the value does not depend on γ, s is smooth since all the γ's are smooth, and s is horizontal, since, by definition $s \circ \gamma(t)$ is the horizontal lift of γ and therefore satisfies $\nabla_{\dot\gamma(t)} s \circ \gamma(t) = 0$, hence $\nabla_X s = 0$. $\square$

The question of whether or not parallel transport is independent of the curve connecting the points in M is essentially related to the notion of curvature which is the subject of the next section.